University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO FINAL EXAMINATION

Problem 1.

a.
We consider this head-on collision in the center of mass. The center of mass velocity is

$$V^* = V\frac{M}{M+m} \approx V$$

Using this approximation, in the C.M. the fly approaches the locomotive with speed V. Since the collision is elastic, it bounces back with the same speed. Transforming back to the lab, the fly has velocity

$$v \approx V + V = 2V$$

b.
In each collision, the momentum $2mV$ that is gained by the fly is lost by the locomotive:

$$\Delta P = M\Delta V = -2mV$$
$$\frac{\Delta V}{V} = -2\frac{m}{M}$$

In a time interval Δt, the volume swept out by the front of the train is $AV\Delta t$; this volume contains $NAV\Delta t$ flies. So, for $NAV\Delta t$ collisions,

$$\frac{\Delta V}{V} = -2\frac{m}{M}NAV\Delta t$$
$$\frac{\Delta V}{V^2} = -2NA\frac{m}{M}\Delta t$$
$$\int\frac{dV}{V^2} = -2NA\frac{m}{M}\int dt$$
$$\frac{1}{V} - \frac{1}{V_0} = 2NA\frac{m}{M}t$$
$$V(t) = \frac{1}{2NA\frac{m}{M}t + \frac{1}{V_0}}$$
$$V(t) = \frac{V_0}{1 + 2NAV_0\frac{m}{M}t}$$

where V_0 is the velocity at $t = 0$.

Problem 2.
At the instant that the probe barely grazes the planet, it will have radius R and velocity $\mathbf{v}_f$ directed tangentially to the planet. Angular momentum conservation requires

$$mv_0b = mv_fR$$
$$v_f = v_0\frac{b}{R}$$

Substituting for v_f in the equation for energy conservation, we obtain

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 - \frac{GM_pm}{R}$$
$$\frac{1}{2}v_0^2 = \frac{1}{2}v_0^2\frac{b^2}{R^2} - \frac{GM_p}{R}$$
$$\frac{1}{2}v_0^2\Big(\frac{b^2}{R^2} - 1\Big) = \frac{GM_p}{R}$$
$$\frac{b^2}{R^2} - 1 = \frac{2GM_p}{v_0^2R}$$
$$b = R\sqrt{1 + \frac{2GM_p}{v_0^2R}}$$

Problem 3.

a.

$$mR\omega^2 = mg$$
$$\omega = \sqrt{\frac{g}{R}}$$

b.

$$2\omega v = a_C = g\frac{v}{v_C}$$
$$v_C = \frac{g}{2\omega}$$
$$v_C = \frac{g}{2}\sqrt{\frac{R}{g}}$$
$$v_C = \frac{1}{2}\sqrt{gR}$$

c.

$$\mathbf{F}_C = -2m(\boldsymbol{\omega} \times \mathbf{v})$$

$\boldsymbol{\omega}$ is north, and $-\mathbf{v}$ is east; $\vec{\text{north}} \times \vec{\text{east}}$ is *down*. This is the direction in which the ball misses.

$$\begin{aligned} a_C &= 2\omega v = 2v\sqrt{\frac{g}{R}} \\ d &= \frac{1}{2}a_C t^2 \\ &= \frac{1}{2}2v\sqrt{\frac{g}{R}}t^2 \\ t &= \frac{D}{v} \\ d &= v\sqrt{\frac{g}{R}}\frac{D^2}{v^2} \\ d &= \frac{D^2}{v}\sqrt{\frac{g}{R}} \end{aligned}$$

(We ignore the centrifugal force on the ball, because it is the same on the colony as on earth, and the pitcher already compensates for it.) As a sanity check, if $D = 20$ m and $v = 40$ m/sec (appropriate to baseball), and $R = 1000$ m, we obtain $d \approx 1$ m. Indeed d is much smaller than D. Nevertheless, from the standpoint of the pitcher, the Coriolis force has a big effect on his control.

Problem 4.
The equation of motion for $x(t)$ is

$$\begin{aligned} m\ddot{x} = -k(x - x_s) &= -m\omega_0^2(x - x_s) \\ \ddot{x} &= -\omega_0^2 x + \omega_0^2 mA\sin\omega t \\ \ddot{x} + \omega_0^2 x &= kA\sin\omega t \end{aligned}$$

a.

$$\begin{aligned} \text{try } x_p(t) &= B\sin\omega t \\ (-\omega^2 + \omega_0^2)B\sin\omega t &= kA\sin\omega t \\ B &= \frac{kA}{\omega_0^2 - \omega^2} \\ x_p(t) &= \frac{kA\sin\omega t}{\omega_0^2 - \omega^2} \end{aligned}$$

b.
Because an infinite force from the spring would be required otherwise, $\dot{x}_0(0) = 0$ as well as $x_0(0) = 0$. The general solution to the homogeneous equation of motion ($A = 0$) is

$$x_h(t) = C\cos\omega_0 t + D\sin\omega_0 t$$

The general solution to the full equation is obtained by adding x_h to x_p. Applying initial conditions,

$$\begin{aligned} x_0(t) &= \frac{kA\sin\omega t}{\omega_0^2 - \omega^2} + C\cos\omega_0 t + D\sin\omega_0 t \\ x_0(0) &= 0 \;\Rightarrow\; C = 0 \\ \dot{x}_0(0) &= 0 \;\Rightarrow\; 0 = \frac{\omega kA}{\omega_0^2 - \omega^2} + \omega_0 D \\ D &= -\frac{\omega}{\omega_0}\frac{kA}{\omega_0^2 - \omega^2} \\ x_0(t) &= kA\frac{\omega_0\sin\omega t - \omega\sin\omega_0 t}{\omega_0(\omega_0^2 - \omega^2)} \end{aligned}$$

Problem 5.

a.

$$\xi(x = 0, t) = \xi(x = L, t) = 0$$

b.

$$\begin{aligned} \xi(x,t) &= \sin kx\,\Re\big(\xi_0\exp(-i\omega t)\big) \\ \sin kL &= 0 \\ kL &= n\pi,\; n = 1, 2, \ldots \\ \omega &\equiv ck \\ \omega_s &= \frac{\pi c}{L} \end{aligned}$$

c.

$$\xi(x,t) = \xi(x + L, t)$$

d.

$$
\begin{aligned}
\xi(x,t) &= \Re\big(\xi_0 \exp\left(i(kx-\omega t)\right)\big) \\
\exp(ikx) &= \exp(ik(x+L)) \\
1 &= \exp(ikL) \\
kL &= 2n\pi,\ n = 1,2,\ldots \\
\omega &\equiv ck \\
\omega_t &= \frac{2\pi c}{L} \\
\omega_t &= 2\omega_s
\end{aligned}
$$

Problem 6.

a.
Per unit mass of fluid, the force $\mathbf{f}$ is

$$\mathbf{f} = -\hat{\mathbf{r}}\frac{GM}{r^2}$$

We seek a function $\Phi(r)$ such that

$$-\nabla\Phi = \mathbf{f}$$

or equivalently, using spherical symmetry,

$$\Phi = -\int f_r dr$$

Clearly

$$\Phi(r) = -\frac{GM}{r}$$

satisfies either of these conditions.

b.
Since the flow is steady, we can use Bernoulli's equation (either along a streamline at constant (θ, ϕ), or, since the flow is irrotational, anywhere outside the black hole):

$$\frac{1}{2}\rho v^2 + p + \rho\Phi = \text{constant}$$

Only the first and third terms are not constant, so they must have the same r dependence. Therefore v^2 and Φ have the same r dependence. So

$$v \propto r^{-1/2}$$

c.
In steady flow there can be no buildup of mass density ρ. Therefore the mass flow

$$\rho v \text{ (kg/m}^2\text{sec)} \times 4\pi r^2 \text{ (m}^2\text{)}$$

through a spherical surface of radius r must be independent of r. So, using the result of part (**a.**),

$$
\begin{aligned}
\rho v &\propto r^{-2} \\
\rho &\propto r^{-3/2}
\end{aligned}
$$

More formally, but equally acceptably, one can reach the same conclusion by applying the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

and using the fact that for steady flow the first term vanishes.